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# Do many-particle neutrino interactions cause a novel coherent effect?

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Abstract: We investigate whether coherent flavor conversion of neutrinos in a neutrino background is substantially modified by many-body effects, with respect to the conventional one-particle effective description. We study the evolution of a system of interacting neutrino plane waves in a box. Using its equivalence to a system of spins, we determine the character of its behavior completely analytically. We find that, if the neutrinos are initially in flavor eigenstates, no coherent flavor conversion is realized, in agreement with the effective one-particle description. This result does not depend on the size of the neutrino wavepackets and therefore has a general character. The validity of the several important applications of the one-particle formalism is thus confirmed.

Keywords: Neutrino and Gamma Astronomy, Neutrino Physics.

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## 1. Introduction

The flavor composition of a neutrino system may be modified as a result of the interactions between the neutrinos and a background medium. The efficiency of this process depends, in addition to the properties of the background itself (composition, density, temperature, etc.), on the coherent (or incoherent) character of the interaction.

Here we study the case of *coherent* neutrino conversion in a background of *neutrinos*. The interaction of a neutrino beam with a neutrino background has the peculiar feature that, due to momentum exchange, a neutrino from the background may be scattered into the beam and vice versa. This differs from the case of scattering on electrons and/or nucleons, and implies that the effect of coherent scattering on neutrinos can not be treated in analogy with that on ordinary matter.

In the literature, the effects of the neutrino-neutrino coherent scattering on neutrino conversion have traditionally been treated following the pioneering work by Pantaleone [1, 2], Sigl & Raffelt [3], and McKellar & Thomson [4]. In these papers, a *single-particle* evolution equation for each neutrino is derived, which has the form of the usual oscillation equation for neutrinos in matter with an extra term designed to capture the effect of neutrino–neutrino scattering. According to this equation, the effect of the interaction with the neutrino background is significant if the neutrino-neutrino potential is at least comparable to the potential due to matter (zero-temperature and thermal terms) and/or to the vacuum oscillation terms:

$$|n_\nu - n_{\bar\nu}| \gtrsim \begin{cases} |n_{e^-} - n_{e^+}| \\ \dfrac{(n_{e^-} + n_{e^+})E_\nu^2}{m_w^2} \\ \dfrac{\Delta m^2}{(G_F E_\nu)} \end{cases} \tag{1.1}$$

Here $n_\nu$ ($n_{\bar\nu}$) is the (anti)neutrino number density, $n_{e^-}$ ($n_{e^+}$) is the electron (positron) number density, $E_\nu$ is the neutrino energy, $m_w$ is the $W$-boson mass and $G_F$ is the Fermi constant.

The condition (1.1) may be satisfied in the early Universe close to the Big Bang Nucleosynthesis epoch (at temperature $T \sim O(1)$ MeV) or just outside a supernova core (at $r \sim O(100$ km$)$). Indeed, in these environments the number density of neutrinos is comparable to, or larger than, that of electrons and nucleons. At the same time, incoherent scattering is negligible and neutrinos stream freely in the medium, affected by coherent scattering (refraction) only. The evolution of the neutrino flavor composition in both cases has been extensively studied using the one-particle equation. An important result [5]–[9] is the strong bound on the chemical potential of non-electron relic neutrinos that follows from studying neutrino oscillations in the early Universe, with important implications on BBN [10]. Other important applications include hypothetical oscillations between active and sterile neutrino states that may have generated a lepton asymmetry in the early Universe [11]–[15] and possible effects of a sterile neutrino on the $r$-processes in a supernova [16]–[20].

There is an important theoretical issue at the foundation of all of the mentioned analyses that, if unresolved, casts doubt on their validity. The issue is the *existence and the range of validity* of the description of neutrino self-refraction in terms of a set of one-particle equations. Generally speaking, the interactions between neutrinos in the ensemble may be expected to create quantum correlations, or, in other words "entangled" states of many neutrinos which are not products of individual wavefunctions. *A priori*, a system in which such entanglement exists requires a many-particle description. The importance of this point was recognized early on [2], but was not pursued further in the following years.

Recently, the problem of possible existence and effects of quantum correlations has been investigated using two different approaches. The first, put forth in our recent paper [21], is to understand the flavor evolution of a many-neutrinos system as an effect of the interference of many elementary neutrino-neutrino scattering events. Using this construction, we argued that, in the limit of infinite number of neutrinos, the many-body description factorizes into one-particle equations. These equations coincide with those given in the previous literature up to terms which change the overall phases of neutrino states. While such terms can be potentially important in non-oscillation phenomena, they do not affect the flavor evolution of the system.



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As a particular example, we considered an ensemble of neutrinos which are initially in flavor eigenstates. Following our analysis, we found that in this system, somewhat counterintuitively, no coherent flavor conversion takes place. This finding is in agreement with the standard one-particle description of [1]–[4].

The second approach, introduced by Bell *et al.* [22], is to consider the evolution of a system of interacting neutrino plane waves in a box. The idea is to study the properties of the many-body hamiltonian of this system with the goal of determining the rate with which statistical equilibration is achieved. The analysis is applied to a system of neutrinos which are initially in flavor eigenstates. It is concluded that the time scale $t_{eq}$ of the flavor evolution of a given neutrino in this system is inversely proportional to the neutrino density and to the Fermi constant: $t_{eq} \propto (nG_F)^{-1}$, which is characteristic of coherent flavor conversion. This result contrasts with the prediction of the one-particle description, and therefore has been considered as an indication of its breakdown and of the existence of new coherent effects due to entanglement. Clearly, this would have profound implications on the applications of neutrino-induced neutrino conversion to supernova physics and cosmology.

Given the importance of the subject, in this paper we present a further study of neutrino flavor conversion in a neutrino background. The aim is to clarify the origin of the contrast between ref. [22] and the other literature and give a definite answer to the question of the importance of entanglement and its effects.

Since the suggested new effect comes from simultaneous interaction between many neutrinos, we study, following [22], the Schrödinger equation of a system of many neutrino plane waves in a box. We consider the free streaming regime and neglect vacuum oscillation terms, which can be included by a straightforward generalization of our results. We show that the problem is equivalent to that of a system of spins, for which, in the case of constant spin-spin coupling, the equilibration time can be determined completely analytically. For neutrinos initially in flavor eigenstates, in the limit of the infinite number of neutrinos, this solution shows no coherent conversion, in agreement with our earlier results.

The paper is organized as follows. In section 2 we lay the foundations of our analysis, by discussing the structure of the neutrino–neutrino interaction in the flavor space and in the real space. We also perform an elementary analysis of coherence in a system of several interacting neutrinos and use it to motivate the following study. In section 3 the many body approaches are discussed and the relevant results of our previous paper are reviewed. In section 4 we formulate the problem of interacting neutrino plane waves in a box, present the full analytical solution and give its derivation. A discussion and conclusions follow in section 5 and 6.

## 2. General considerations

### 2.1 The flavor structure of the interaction

In the range of neutrino energies that are relevant to the applications ($\sim 10^6 - 10^7\,\mathrm{eV}$), the interaction between neutrinos is described by the low-energy neutral current (NC) hamiltonian

$$H_{\rm NC} = \frac{G_F}{\sqrt{2}} \left( \sum_a \bar{\nu}_a \gamma^\mu \nu_a \right) \left( \sum_b \bar{\nu}_b \gamma_\mu \nu_b \right) . \tag{2.1}$$

Higher order effects, such as those coming from the expansion of the propagator or particle production (bremsstrahlung) will be neglected in our analysis.

A property of this hamiltonian that will play a very important role later is its invariance under the rotation of the SU(2) flavor group. Because of this property, the interaction between pairs of neutrinos — viewed in flavor space — must be equivalent to the interaction between pairs of spins.

The one-particle equations of [1]–[4, 21] indeed preserve the SU(2) structure of the problem and have a form of the interaction between spins.[1] It would be logically inconsistent, however, to rely on this fact in the analysis, in which these equations are being tested. Therefore, below we present an explicit proof of the equivalence, valid regardless of whether the interaction is coherent or not.

Let us consider the interaction of two neutrinos, and, in the interests of clarity, for a moment suppress the Dirac indices on the fermions and the gamma matrices. Since the NC interactions conserve flavor, in the hamiltonian (2.1) the flavor space wavefunction of a given outgoing neutrino is equal to that of one of the incoming neutrinos. Let us denote the flavor wavefunctions by $\Psi_i$ and $\Phi_i$. There are then two possible combinations:

$$\Psi_i^* \delta_{ij} \Psi_j \Phi_k^* \delta_{kl} \Phi_l \, , \tag{2.2}$$

$$\Psi_i^* \delta_{il} \Phi_l \Phi_k^* \delta_{kj} \Psi_j \, . \tag{2.3}$$

Since $\sum_{i=1}^2 |\Psi_i|^2 = \sum_{i=1}^2 |\Phi_i|^2 = 1$, the term in eq. (2.2) equals 1. The term in eq. (2.3) could be transformed using the well-known property of the (4-dimensional) $\sigma$-matrices, $\sigma^m_{\alpha\dot{\alpha}} \bar{\sigma}_m^{\dot{\beta}\beta} = -2\delta_\alpha^\beta \delta_{\dot{\alpha}}^{\dot{\beta}}$ (in the notation convention of [24]), or

$$2\delta_{il}\delta_{jk} = \delta_{ij}\delta_{kl} + \vec{\sigma}_{ij} \cdot \vec{\sigma}_{kl} \, . \tag{2.4}$$

One gets

$$\Psi_i^* \delta_{il} \Phi_l \Phi_j^* \delta_{jk} \Psi_k = \frac{1}{2} (1 + \Psi_i^* \vec{\sigma}_{ij} \Psi_j \cdot \Phi_k^* \vec{\sigma}_{kl} \Phi_l) \, . \tag{2.5}$$

In this form, the equivalence between a system of neutrinos and a system of spins is manifest. The complete (including both the contributions from (2.5) and (2.2)) flavor space hamiltonian for the interaction of two neutrinos, 1 and 2, is proportional to

$$\left( \frac{3}{2} + \frac{\vec{\sigma}_1 \cdot \vec{\sigma}_2}{2} \right) , \tag{2.6}$$

which coincides with the square of the operator of the *total* angular momentum, $\hat{L}^2 \equiv (\vec{\sigma}_1/2 + \vec{\sigma}_2/2)^2$, as expected.

[1] In [23], this property of the equations has been used to give an elegant physical analogy designed to explain the "synchronized" oscillations of neutrinos in a neutrino background.

The strength of the interaction depends on the scattering angle in real space. Let us denote the states of the incoming neutrinos as $\chi_\alpha$ and $\psi_\beta$ and those of the outgoing states as $\bar{\psi}'_{\dot{\beta}}$ and $\bar{\chi}'_{\dot{\alpha}}$. The spatial dependence of the scattering amplitude is given by

$$\bar{\psi}'_{\dot{\beta}}\bar{\sigma}^{\dot{\beta}\beta m}\psi_\beta\bar{\chi}'_{\dot{\alpha}}\bar{\sigma}_m^{\dot{\alpha}\alpha}\chi_\alpha = 1 - (\bar{\psi}'\vec{\sigma}\psi)\cdot(\bar{\chi}'\vec{\sigma}\chi)\,. \tag{2.7}$$

If the neutrino wavepackets are sufficiently broad so that several wavepackets overlap, the preceding argument can be trivially generalized to show that the interactions between the "overlapping" neutrinos have the structure of the interactions between the corresponding spins. In the case of neutrino plane waves in a box, each spin interacts with all the others and the total hamiltonian is the sum of the interactions of all pairs (see section 4.3).

### 2.2 Coherence of neutrino-neutrino scattering: basic features

We next discuss general properties of coherent neutrino-neutrino scattering.

By definition, scattering is coherent when the waves scattered by different particles in a target interfere with each other. When the scatterer is distinguishable from the incident particle, the coherence condition is satisfied when the incident particle is scattered forward. For other directions, scattering is incoherent (unless the particle spacings in the target satisfy particular conditions, e.g. periodicity). In contrast, for neutrino-neutrino scattering, scattering can be coherent when the momentum of the scattered neutrino coincides with the momentum of *one* of the incident neutrinos. The corresponding values of the scattering angle $\beta$ are 0 and $\Theta$, where $\Theta$ denotes the angle between the two initial-state neutrinos.

For the purpose of studying coherent effects, one may replace the full hamiltonian (2.1) with a "toy" hamiltonian which restricts scattering angles to values 0 and $\Theta$ only, for which coherent interference may occur. This simplified interaction can be viewed as a flavor exchange between neutrinos which do not change their momenta. We emphasize that, while all coherent effects in the system are fully captured in this way, most of the incoherent effects are left out. Hence, one must be careful while interpreting incoherent effects found in this framework, as will be seen later.

Following [2], we write the reduced hamiltonian for a pair of interacting neutrinos with momenta $\vec{k}$ and $\vec{p}$ in the form

$$i\frac{d}{dt}\begin{pmatrix}|\nu_e(\vec{k})\nu_e(\vec{p})\rangle\\|\nu_e(\vec{k})\nu_\mu(\vec{p})\rangle\\|\nu_\mu(\vec{k})\nu_e(\vec{p})\rangle\\|\nu_\mu(\vec{k})\nu_\mu(\vec{p})\rangle\end{pmatrix} = \frac{\sqrt{2}G_F}{V}(1-\cos\Theta)\begin{pmatrix}2&0&0&0\\0&1&1&0\\0&1&1&0\\0&0&0&2\end{pmatrix}\begin{pmatrix}|\nu_e(\vec{k})\nu_e(\vec{p})\rangle\\|\nu_e(\vec{k})\nu_\mu(\vec{p})\rangle\\|\nu_\mu(\vec{k})\nu_e(\vec{p})\rangle\\|\nu_\mu(\vec{k})\nu_\mu(\vec{p})\rangle\end{pmatrix}, \tag{2.8}$$

where $V$ is the normalization volume. The value of the angular factor, $(1-\cos\Theta)$, follows from (2.7) (taking into account the anticommutativity of the spinors), while the flavor structure of the interaction is determined by eq. (2.6).

The condition $\beta = \Theta$ by itself, however, *does not guarantee* coherence. We can see that by considering the scattering of an electron neutrino on two "background" muon neutrinos. Suppose that the initial momenta of the muon neutrinos are orthogonal to that of the electron neutrino and, further, that the neutrino wave packets are sufficiently narrow so that the two scattering events are independent, with each given by eq. (2.8). Then, as a result of the interaction, we get

$$|\nu_e\rangle|\nu_\mu\nu_\mu\rangle \longrightarrow |F\rangle = (1+2ia)|\nu_e\rangle|\nu_\mu\nu_\mu\rangle + ia(|\nu_\mu\rangle|\nu_e\nu_\mu\rangle + |\nu_\mu\rangle|\nu_\mu\nu_e\rangle)\,, \tag{2.9}$$

where $a \equiv -\sqrt{2}G_F(1-\cos\Theta)t/V$.

Both the interaction time $t$ and the volume $V$ depend on the size of the wavepackets. Taking the wavepackets for simplicity to be spheres of size $l$, we can write

$$t \sim l\,, \qquad V \sim l^3\,, \tag{2.10}$$

$$a \sim -\frac{G_F(1-\cos\Theta)}{l^2}\,. \tag{2.11}$$

We notice that $|a| \ll 1$, as long as the size of the wavepackets is much greater than $(100\,\mathrm{GeV})^{-1}$, i.e., the neutrino energy is well below the weak scale. This justifies neglecting terms of order $a^2$ and higher in eq. (2.9).

Let us compute the probability $P$ that, as a result of the interaction, the incident neutrino was converted into $\nu_\mu$. Introduce the "$\nu_\mu$ number" operator $\hat{\mu} \equiv |\mu\rangle\langle\mu|$ that acts only on the state of the incident neutrino. Using the orthogonality of the states $|\nu_e\nu_\mu\rangle$ and $|\nu_\mu\nu_e\rangle$, the probability in question is

$$P = \langle F|\hat{\mu}|F\rangle = 2a^2\,. \tag{2.12}$$

This shows that the process is *incoherent*: the probability we found coincides with the sum of the conversion probabilities for each scattering (a coherent process would yield $P = (2a)^2$). As can be easily confirmed, as the number of "background" neutrinos $N$ is increased, the probability of flavor conversion goes as $Na^2$.

It can be further shown that if the background neutrinos are in a flavor superposition state $\nu_x = \cos\alpha\nu_e + \sin\alpha\nu_\mu$, the probability of conversion is *not*, in general, equal to the sum of the probabilities for each scattering. Indeed, a straightforward calculation yields $P = a^2\sin^2\alpha(2+2\cos^2\alpha)$. Comparing this with the conversion probability for a single scattering event, $a^2\sin^2\alpha$, we see that the conversion process is *partially coherent*, and the degree of coherence increases with decreasing $\alpha$. A generalization to a larger number of scatterers is [21]

$$P = a^2\sin^2\alpha((N^2-N)\cos^2\alpha + N)\,, \tag{2.13}$$

which clearly shows the presence of partial coherence (the terms proportional to $N^2$).

The above example shows that whether a neutrino undergoes coherent conversion or not depends on the flavor state of the many background neutrinos it interacts with. Moreover, as can be seen from eq. (2.9), each neutrino-neutrino interaction creates an entangled state. Hence, the flavor evolution of a system of several interacting neutrinos demands a many-particle description. The question is: does the wavefunction of the system somehow factorize into individual one-particle wavefunctions in the limit of large-$N$, and, if yes, under what conditions? Below we describe two recently proposed approaches to this problem.



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## 3. Single- vs. many-particle description: two approaches

### 3.1 First approach: interference of many elementary scattering amplitudes

The first method was developed in [21]. The idea is to describe the flavor evolution in a neutrino gas by generalizing the procedure of adding elementary scattering amplitudes, as was outlined in the three-neutrino example of section 2.2.

Let us consider the interactions between two orthogonal neutrino beams. Once the results for this setup are understood, they can be straightforwardly generalized to more complicated systems. Let us assume that the neutrino wavepacket size is much smaller than the particle spacing ($l \ll n_\nu^{-1/3}$), so that different neutrino-neutrino interactions can be treated independently using eq. (2.8). As an initial configuration, we take the first beam to be made of $N_1$ electron neutrinos $\nu_e$, and the second beam to be made of $N_2$ neutrinos in the flavor superposition state $\nu_x \equiv \cos\alpha\nu_e + \sin\alpha\nu_\mu$. It can be shown [21] that the result of the interaction during a small time $\delta t$ can be written as

$$
\begin{aligned}
|eee\cdots\rangle|xxx\cdots\rangle \stackrel{t\to t+\delta t}{\longrightarrow} |F\rangle &= |eee\cdots\rangle|xxx\cdots\rangle + ia|F_1\rangle\,, &(3.1)\\
|F_1\rangle &= N_1N_2(1+|\langle e|x\rangle|^2)|eee\cdots\rangle|xxx\cdots\rangle + \\
&\quad +N_2\langle\mu|x\rangle\langle x|e\rangle(|\mu ee\cdots\rangle + |e\mu e\cdots\rangle+\cdots)|xxx\cdots\rangle + \\
&\quad +N_1\langle e|x\rangle\langle y|e\rangle|eee\cdots\rangle(|yxx\cdots\rangle+|xyx\cdots\rangle+\cdots) + \\
&\quad +\langle\mu|x\rangle\langle y|e\rangle(|\mu ee\cdots\rangle+|e\mu e\cdots\rangle+\cdots)\times \\
&\quad \times(|yxx\cdots\rangle+|xyx\cdots\rangle+\cdots)\,, &(3.2)
\end{aligned}
$$

a direct generalization of eq. (2.9). Here the state $|y\rangle$ is defined to be orthogonal to $|x\rangle$: $\langle x|y\rangle = 0$.

The evolution in eqs. (3.1), (3.2) cannot be described using only one-particle equations, since the final state in eq. (3.2) is an entangled many-particle state. Nevertheless, as observed in [21], *the coherent part* of the evolution can be. Indeed, the last term in eq. (3.2) represents an incoherent effect, since it contains the sums of mutually orthogonal terms, and could be dropped. The sum of the remaining three terms is equal, to the first order in $a$, to a product of rotated single-particle states.

Since for small $\delta t$ a state of each neutrino undergoes a small rotation and since nothing in this argument depends on the particular choice of the initial states, eqs. (3.1), (3.2) specify how the states will rotate at any point in time. This observation means that, for each neutrino, one can write a differential equation describing its evolution for any — not necessarily small — $t$. This equation takes on the form [21]

$$
\begin{aligned}
i\frac{d|\psi^{(i)}\rangle}{dt} &= H_{aa}|\psi^{(i)}\rangle\,,\\
H_{aa} &= \sum_j \sqrt{2}G_F n^{(j)}(1-\cos\Theta^{(ij)})\left[|\phi^{(j)}\rangle\langle\phi^{(j)}| + \frac{1}{2} - \frac{1}{2}|\langle\phi^{(j)}|\psi^{(i)}\rangle|^2\right], \qquad (3.3)
\end{aligned}
$$

where, for generality, we restored the angular factor and summed over possible angles $\Theta^{(ij)}$. The flavor-space wavefunction of the given neutrino is denoted by $\psi^{(i)}$, those of the other ("background") neutrinos by $\phi^{(j)}$. The quantity $n^{(j)}$ denotes the number density of neutrinos whose momenta make an angle $\Theta^{(ij)}$ with the given neutrino. This equation is similar to the one-particle evolution equations given in [1, 3, 4]. In fact, the only difference is in the last two terms in the square brackets, which have no effect on flavor evolution.

Let us discuss a particular case when the neutrinos at $t = 0$ are in the flavor eigenstates. We note that, according to eq. (3.3), a coherent conversion effect in this case is absent. This result in turn could be traced to eq. (3.2), where the second and the third terms vanish for $\nu_x = \nu_\mu$ or $\nu_x = \nu_e$. The only remaining source of flavor transformation is the last term in eq. (3.2), which is nonzero for $\nu_x = \nu_\mu$. Since this term represents incoherent scattering, for a neutrino traveling through a gas of neutrinos of opposite flavor the conversion probability for small $t$ has the form $P(t) = Nta^2$.

At first, it is not obvious how this incoherent effect is related to the incoherent flavor exchange that occurs in a real neutrino ensemble. Indeed, as mentioned in the introduction to eq. (2.8), by replacing the true interaction eq. (2.1) with the forward scattering hamiltonian, we have truncated most of the incoherent effects. Nevertheless, a connection between the two can be made. The argument, which follows below, is very instructive and will prove helpful in the subsequent sections.

When discussing the "forward scattering probability", we in fact mean the probability for the neutrino to be scattered in a certain (small) solid angle $\delta\Omega$ around the direction of the incoming neutrino (the probability of neutrino scattering at any fixed angle is, of course, zero). A neutrino wave packet with the cross section area $A$ and wavelength $\lambda = 2\pi/E_\nu$ is diverging due to diffraction into a solid angle $\delta\Omega_{\rm diff} = \lambda^2/A$. The neutrino wave which is scattered into the same solid angle $\delta\Omega$ will interfere with the incident wave. Thus, $\delta\Omega = \delta\Omega_{\rm diff}$.

With this in mind, let us consider the result of forward scattering of a given neutrino on $N$ other neutrinos in the medium. According to eq. (2.8), for neutrino wave packets of longitudinal size $l$ and cross section $A$, $a \sim -G_F l(1-\cos\Theta)/Al = -G_F(1-\cos\Theta)/A$. The resulting flavor conversion probability $P$ depends on whether the forward scattering amplitudes add up coherently or incoherently. In the first case, $P_{\rm coh} \sim N^2a^2$, in the second, $P_{\rm inc} \sim Na^2$. The number of interactions during time $t$ is $N = Atn_\nu$ ($n_\nu$ is the neutrino number density) and we find that in the case of coherent forward scattering the dependence on the size of the wave packet *drops out*,

$$P_{\rm coh} \sim \frac{G_F^2(1-\cos\Theta)^2}{A^2}A^2t^2n_\nu^2 = G_F^2(1-\cos\Theta)^2t^2n_\nu^2\,. \tag{3.4}$$

This is indeed seen from eq. (3.3), which does not contain the size of the neutrino wavepacket.

On the other hand, for the case of incoherent forward scattering, the cross section of the wave packet does enter the final result,

$$P_{\rm inc} \sim \frac{G_F^2(1-\cos\Theta)^2}{A^2}Atn_\nu = \frac{G_F^2(1-\cos\Theta)^2}{A}tn_\nu\,. \tag{3.5}$$

The dependence on the size of the wavepacket look puzzling. Nevertheless, it is easily understood once we recall that, by construction, the result in (3.5) represents the fraction



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of incoherent scattering events for which one of the two neutrinos is scattered in the forward cone $\delta\Omega \sim \lambda^2/A \propto 1/E_\nu^2 A$. This means that, up to a numerical coefficient, the probability of incoherent scattering in any direction $P_{\rm inc}^{\rm TOT}$ is given by

$$P_{\rm inc}^{\rm TOT} \sim \frac{P_{\rm inc}}{\delta\Omega} \sim G_F^2 E_\nu^2 (1-\cos\Theta)^2 t n_\nu \,. \tag{3.6}$$

This has the form $\sigma_w t n_\nu$, where $\sigma_w$ is a typical weak interaction cross section, $\sigma_w \propto G_F^2 E_\nu^2$, exactly as one would expect.

Thus, in summary, for the purpose of studying coherent effects, the full hamiltonian (2.1) may be replaced by the forward scattering flavor exchange hamiltonian, as done in eq. (2.8), with the understanding that in the final result the terms that go like $N^2$ represent the *complete* coherent effect while the terms that go like $N$ correspond to only the *fraction* of the incoherent scattering that occurs in the forward diffraction cone.

### 3.2 Second approach: solving a many-particle equation for neutrino plane waves in a box

An alternative method of studying coherent processes in a neutrino gas was proposed in [22]. The idea is to consider a system of interacting neutrino plane waves in a box and study its flavor evolution, again with the restriction to forward scattering only. The neutrinos are taken to be initially in flavor states and the goal is to test the existence of coherent conversion effects by looking at the rate of equilibration of the system.

The "forward scattering" hamiltonian can be found by generalizing eq. (2.8). A basis of states is formed by the initial configuration of neutrinos and all its possible permutations. Taking, e.g., $N$ electron and $N$ muon neutrinos, there are $(2N)!/(N!)^2$ distinct basis states. The entries of the hamiltonian are found as follows. (i) The diagonal entries receive $N(3N-2)$ contributions (this includes $(2N)!/[2!(2N-2)!] = N(2N-1)$ flavor-blind scattering processes with scattering angle $\beta = 0$ and $2N!/[2!(N-2)!] = N(N-1)$ processes with scattering angle $\beta = \Theta$). (ii) Each off-diagonal entry receives a single contribution from the exchange process that connects the two corresponding basis states; if the two basis states are not connected by a single permutation, the corresponding off-diagonal entry vanishes. (iii) Each contribution equals $\sqrt{2}G_F(1-\cos\Theta_{ij})/V$, where $\Theta_{ij}$ is the angle between the momenta of the two interacting neutrinos.

For example, for the case of $N = 2$ there are 6 basis states: $|\nu_e\nu_e\nu_\mu\nu_\mu\rangle$, $|\nu_\mu\nu_e\nu_e\nu_\mu\rangle$, $|\nu_\mu\nu_e\nu_\mu\nu_e\rangle$, $|\nu_e\nu_\mu\nu_e\nu_\mu\rangle$, $|\nu_e\nu_\mu\nu_\mu\nu_e\rangle$, $|\nu_\mu\nu_\mu\nu_e\nu_e\rangle$, and the hamiltonian is given by

$$H_{N=2} = \frac{\sqrt{2}G_F}{V}\begin{pmatrix} d_{12,34} & f_{13} & f_{14} & f_{23} & f_{24} & 0 \\ f_{13} & d_{14,23} & f_{34} & f_{12} & 0 & f_{24} \\ f_{14} & f_{34} & d_{13,24} & 0 & f_{12} & f_{23} \\ f_{23} & f_{12} & 0 & d_{13,24} & f_{34} & f_{14} \\ f_{24} & 0 & f_{12} & f_{34} & d_{14,23} & f_{13} \\ 0 & f_{24} & f_{23} & f_{14} & f_{13} & d_{12,34} \end{pmatrix}. \tag{3.7}$$

Here $f_{ij} = 1-\cos\Theta_{ij}$ is the angular factor for the $i,j$ pair and $d_{ij,kl} = f_{ij} + f_{kl} + \sum$, where $\sum$ is the sum of all distinct $f_{ij}$.

Let us discuss the meaning of the equilibration time $t_{\rm eq}$. If conversion is coherent, $t_{\rm eq}$ is defined as the time over which a neutrino of a given momentum is expected to change its flavor. The momentum itself stays unchanged over this time scale. If conversion is incoherent, $t_{\rm eq}$ is defined as the time over which the neutrino momenta will be randomized (each neutrino can no longer be identified by its momentum). In the model we are considering, however, the definition of equilibration for coherent process also applies to the incoherent process, since the interaction is chosen to preserve the neutrino momentum.

Let us determine what dependence of $t_{\rm eq}$ would be a sign of coherent or incoherent conversion. To do this, it is useful to repeat the exercise of section 3.1 of estimating the conversion probability as a function of time. For neutrino plane waves in the box the following two modifications to the argument need to be made: first, the normalization volume $V$ becomes the size of the box and, second, the interaction time becomes the actual time $t$ elapsed from the beginning of the evolution. The scattering amplitude becomes $a \sim -G_F t(1-\cos\Theta)/V$ and, since each neutrino simultaneously interact with all the others, the number of interactions is $N = V n_\nu$. Using these ingredients, we find

$$P_{\rm coh} \sim \frac{G_F^2(1-\cos\Theta)^2 t^2}{V^2} V^2 n_\nu^2 = G_F^2(1-\cos\Theta)^2 t^2 n_\nu^2\,, \tag{3.8}$$

$$P_{\rm inc} \sim \frac{G_F^2(1-\cos\Theta)^2 t^2}{V^2} V n_\nu = \frac{G_F^2(1-\cos\Theta)^2}{V} t^2 n_\nu\,. \tag{3.9}$$

Here, as in the case of the interacting neutrino wavepackets, we find that in $P_{\rm coh}$ the dependence on the size of the box cancels out (and the result is the same as eq. (3.4)), while for incoherent scattering the size of the box enters the final result.

Inverting eqs. (3.8), (3.9) and eqs. (3.4), (3.5), we find that the equilibration time for coherent scattering in both cases has the dependence

$$t_{\rm eq}^{coh} = (G_F n_\nu)^{-1}\,, \tag{3.10}$$

while for incoherent scattering one finds

$$t_{\rm eq}^{inc} = \left(\frac{G_F^2 n_\nu}{A}\right)^{-1} \tag{3.11}$$

for the case of neutrino wavepackets, and

$$t_{\rm eq}^{inc} = (G_F\sqrt{n_\nu/V}\,)^{-1} \tag{3.12}$$

for plane waves in a box.

It is important to stress that the expression for $t_{\rm eq}^{coh}$ is the same in both pictures, while the expressions for $t_{\rm eq}^{inc}$ are different. The main reason for the difference has to do with the fact that our "forward scattering" hamiltonian (2.8) captures the entire coherent effect, but only a fraction of the incoherent effect, as already discussed.[2] Eq. (3.10), thus, gives the

[2]Another, more subtle, reason has to do with the fact that, although each wavepacket is a superposition of plane waves, *the time structure* of the interaction between two wavepackets is not the same as the interaction between plane waves. In particular, we get a $t^1$ dependence in eq. (3.5), but a $t^2$ dependence in eq. (3.9).

true coherent equilibration time one would expect in a physical system, while eqs. (3.11) and (3.12) give the answers for a fictitious problem with the hamiltonian (2.8).

In what follows, we solve the evolution of the system of neutrinos in the box interacting with the hamiltonian (2.8) completely analytically. This gives the equilibration time $t_{eq}$, which we compare to eqs. (3.10) and (3.12) to determine the character of the evolution.

## 4. Plane waves in a box: analytical treatment

### 4.1 Setup

As done in [22], we consider a system of $2N$ neutrino plane waves, of which $N$ are initially in the $\nu_e$ state and the remaining $N$ are in the $\nu_\mu$ state. We write this configuration of the system as:

$$|S(0)\rangle = |\underbrace{\nu_e\nu_e\cdots\nu_e}_{N}\underbrace{\nu_\mu\nu_\mu\cdots\nu_\mu}_{N}\rangle\,. \tag{4.1}$$

Here each neutrino wave is identified by its position in the list, which corresponds to a given momentum.

The hamiltonian for this system has the form described in section 3.2. Since it is equivalent to the hamiltonian of a system of spins (section 2.1), in the following we adopt the terminology of angular momenta. As a convention, we identify the "up" (pointing along the $z$ axis) state of a spin with $\nu_e$ and the "down" state with $\nu_\mu$. Thus, the initial state (4.1) can be expressed as

$$|S(0)\rangle = \left|\underbrace{+\frac{1}{2},+\frac{1}{2},\cdots+\frac{1}{2}}_{N},\underbrace{-\frac{1}{2},-\frac{1}{2},\cdots-\frac{1}{2}}_{N}\right\rangle\,. \tag{4.2}$$

To make an analytical treatment possible, we set

$$f_{ij} = 1 \tag{4.3}$$

for all neutrino pairs. The resulting hamiltonian describes an ensemble of spins interacting with each other with equal strength.[3] This simplifying assumption does not change the coherent (or incoherent) character of the conversion effects, as also pointed out in [22]. With this assumption, the nonzero off-diagonal entries of the hamiltonian become equal to $\sqrt{2}G_F/V$ while the diagonal entries become $\sqrt{2}G_F/V \times N(3N-2)$.

We recall (section 3.1) that for the initial configuration (4.1) the single-particle formalism predicts no coherent effects, as can be seen from eq. (3.3). We investigate if this conclusion is changed once the simultaneous interactions between all neutrinos are considered.

### 4.2 Results

Let us first display our results. Their derivation and a more detailed discussion is given in section 4.3.

[3] This makes it different from the familiar Ising model.



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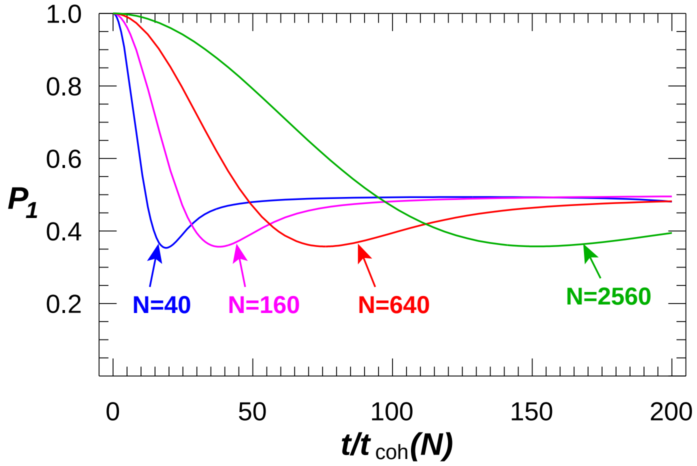


**Figure 1:** The time-evolution of the survival probability $P_1$, for different values of $N$ (numbers on the curves). The unit of the horizontal axis is chosen to be $t/t_{eq}^{coh} = 2tgN$, as used in [22]. If the conversion were coherent, the curves should reach equilibration at the same point on this scale.

The probability $P_1$ that the first spin — which is in the state $|+1/2\rangle$ at $t = 0$ — is found in the same up state at time $t$ ($t > 0$) can be calculated by a completely analytical procedure. It is given by the Fourier series:

$$P_1(t) = \frac{1}{2} + \frac{1}{2} \sum_{J=0}^{N-1} \eta(N, J) \cos[(2J + 2)gt], \tag{4.4}$$

where $g \equiv \sqrt{2} G_F / V$. The coefficients $\eta(N, J)$ are the squares of the Clebsch-Gordan coefficients $\langle j_1 m_1 j_2 m_2 | JM \rangle \equiv C(j_1, m_1, j_2, m_2, J, M)$ for the addition of two angular momenta, $j_1 = m_1 = (N-1)/2$ and $j_2 = -m_2 = N/2$:

$$\begin{aligned} \eta(N, J) &= \left[ C\left( \frac{N-1}{2}, \frac{N-1}{2}, \frac{N}{2}, -\frac{N}{2}, J + \frac{1}{2}, -\frac{1}{2} \right) \right]^2 \\ &= \frac{(1+2J)N[(N-1)!]^2}{(J+N+1)!(N-J-1)!} \,. \end{aligned} \tag{4.5}$$

We follow the standard notation, in which $j_1, j_2$ denote the values of the angular momenta being added, $m_1, m_2$ denote their projections along the $\hat{z}$ direction, and $J$ and $M$ are the corresponding values for the total spin.

Figure 1 shows the survival probability $P_1(t)$, obtained by summing numerically the series (4.4). The horizontal axis represents time in units of the equilibration time predicted for coherent conversion, $t_{eq}^{coh} = (2Ng)^{-1}$ (eq. (3.10) for $2N$ neutrinos). The curves refer to $N = 40, 160, 640, 2560$. Having the same choice of units, our plot can be directly compared to figure 2 of ref. [22], where, however, smaller values of $N$ were used.[4] From figure 1, it is clear that the equilibration time $t_{eq}$ is larger than $t_{eq}^{coh}$ and the discrepancy grows with $N$.

[4]In ref. [22] a different form of the spin-spin (neutrino-neutrino) coupling was adopted, namely, the angular factors were randomly generated. This produces a numerical difference with respect to our results, but does not affect the conclusions about the equilibration time.



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In the limit $N \gg 1$ the series (4.4) is well approximated by the following function:

$$P_1(t) = 1 - \frac{\sqrt{\pi}}{2}\sqrt{N}gt\exp(-Ng^2t^2)\mathrm{erfi}(\sqrt{N}gt), \tag{4.6}$$

where $\mathrm{erfi}(z)$ is the imaginary error function

$$\mathrm{erfi}(z) \equiv -i\ \mathrm{erf}(iz) = \frac{2}{\sqrt{\pi}} \times \int_0^z \exp(t^2)dt\,. \tag{4.7}$$

We have checked that for the values of $N$ shown in figure 1 the approximation is very good and the curves obtained using (4.6) are indistinguishable from those obtained from eq. (4.4).

From eq. (4.6) we see that the characteristic equilibration time equals:

$$t_{eq} = (g\sqrt{N}\,)^{-1}\,. \tag{4.8}$$

This scaling of the equilibration time can be clearly seen in figure 1. It is precisely (up to numerical factors) the form (3.12) expected if the equilibration process is incoherent.

### 4.3 Derivation of results

#### 4.3.1 The eigenvalues of the hamiltonian

From eqs. (2.6) and (4.3) it follows that for each pair $i$ and $j$ the interaction is given by $H_{ij} = g\hat{L}_{ij}^2 = g(\vec{\sigma}_i + \vec{\sigma}_j)^2$ and, therefore, the hamiltonian for the whole system is

$$H_N = g\sum_{i=1}^{2N-1}\sum_{j=i+1}^{2N}(\vec{\sigma}_i + \vec{\sigma}_j)^2 = g\left(\sum_{i=1}^{2N-1}\sum_{j=i+1}^{2N} 2\vec{\sigma}_i\vec{\sigma}_j + \frac{3}{2}N(2N-1)\right), \tag{4.9}$$

where the summation is done over all $N(2N-1)$ pairs.

We notice that this hamiltonian can be related to the operator of the square of the total angular momentum of the system,

$$\hat{L}^2 \equiv \left(\sum_{i=1}^{2N}\vec{\sigma}_i\right)^2 = \sum_{i=1}^{2N-1}\sum_{j=i+1}^{2N} 2\vec{\sigma}_i\vec{\sigma}_j + \frac{3}{2}N\,. \tag{4.10}$$

By comparing eqs. (4.9) and (4.10) we find

$$H_N = g[\hat{L}^2 + 3N(N-1)]\,, \tag{4.11}$$

which has the eigenvalues

$$E_{N,J} = g[J(J+1) + 3N(N-1)]\,, \qquad \text{with} \qquad J = 0, 1, \ldots, N\,. \tag{4.12}$$

Since the flavor evolution will depend only on the difference of the eigenvalues (as will be shown later), in the following we will omit the second term in eq. (4.12) and write

$$E_J = gJ(J+1)\,. \tag{4.13}$$

A given value of $E_J$ in general corresponds to more than one state. This degeneracy can be seen by simply counting the states. For our specific setup, with $M = 0$, there are $(2N)!/(N!)^2$ basis states, while there are only $N+1$ different values of $J$. In general, therefore, an orthonormal basis of eigenstates of the hamiltonian has the form:

$$|J, k_J\rangle \,, \tag{4.14}$$

where the index $k_J$ labels eigenstates with the same value of $J$.

As an illustration of the present discussion, we give the hamiltonian for the case $N = 2$ of section 3.2:

$$H_{N=2} = g \begin{pmatrix} 8 & 1 & 1 & 1 & 1 & 0 \\ 1 & 8 & 1 & 1 & 0 & 1 \\ 1 & 1 & 8 & 0 & 1 & 1 \\ 1 & 1 & 0 & 8 & 1 & 1 \\ 1 & 0 & 1 & 1 & 8 & 1 \\ 0 & 1 & 1 & 1 & 1 & 8 \end{pmatrix} . \tag{4.15}$$

The eigenvalues, which are degenerate as expected, are $g(6, 6, 8, 8, 8, 12)$. They correspond to the following values of $J$: $(0, 0, 1, 1, 1, 2)$, as can be seen from eq. (4.12).

### 4.3.2 The Fourier series

Let us start with proving eq. (4.4).

In the interest of clarity, we first display some definitions. Let us denote as $|+1/2\rangle_1$ and $|-1/2\rangle_1$ the "up" and "down" states of the first spin. Given a state $|\psi\rangle$ of the $2N$ spins of the system, it will be convenient to use the projections $\langle +1/2|_1|\psi\rangle$ and $\langle -1/2|_1|\psi\rangle$, which are states of $2N-1$ spins. In particular, we introduce the state $|S^-\rangle$ obtained by removing the first spin from the initial configuration $|S(0)\rangle$ (eq. (4.2)):

$$\langle +1/2|_1|S(0)\rangle \equiv |S^-\rangle \equiv \left| \underbrace{+\frac{1}{2}, +\frac{1}{2}, \cdots + \frac{1}{2}}_{N-1}, \underbrace{-\frac{1}{2}, -\frac{1}{2}, \cdots - \frac{1}{2}}_{N} \right\rangle . \tag{4.16}$$

For any state, we can define another state obtained from the first one upon a reflection along the $z$-axis. In this transformation, each constituent spin $|+1/2\rangle$ is transformed into spin $|-1/2\rangle$ and vice versa. For instance, the application of this transformation to $|S^-\rangle$ gives its "mirror" state:

$$|S^+\rangle \equiv \left| \underbrace{-\frac{1}{2}, -\frac{1}{2}, \cdots - \frac{1}{2}}_{N-1} \underbrace{+\frac{1}{2}, +\frac{1}{2}, \cdots + \frac{1}{2}}_{N} \right\rangle . \tag{4.17}$$

In what follows we will exploit the properties of various states under this reflection.

We adopt the notation $|j, m, s\rangle_n$ to indicate the result of the following procedure: consider the state $|s\rangle$ and take its projection on the subspace formed by the states of $n$ spins that have the total angular momentum $j$ and projection $m$ along the $\hat{z}$ axis; normalize the result to 1. As an example, $|J, 0, S(0)\rangle_{2N}$ indicates the normalized projection of $S|(0)\rangle$ on the subspace of the states of $2N$ spins having total angular momentum $J$ and $\hat{z}$-projection equal to zero.

It is also convenient to define the state:

$$|s_J^-\rangle \equiv \langle +1/2|_1 |J, 0, S(0)\rangle_{2N} , \tag{4.18}$$

(which has $2N-1$ spins) and denote as $|s_J^+\rangle$ its mirror state obtained upon $\hat{z}$-reflection.

The proof is constituted by four parts.

1. *Expansion of the initial state.* The initial state can be expanded on the basis of the eigenstates of the hamiltonian:

$$|S(0)\rangle = \sum_{J,k_J} |J, k_J\rangle\langle J\,, k_J|S(0)\rangle, \tag{4.19}$$

which then can be evolved in time as

$$|S(t)\rangle = \sum_{J,k_J} e^{-iE_J t}|J, k_J\rangle\langle J, k_J|S(0)\rangle\,. \tag{4.20}$$

Since the energy depends only on $J$, the summation over $k_J$ can be done, yielding

$$\begin{aligned} |S(0)\rangle &= \sum_{J=0}^{N} A_J |J, 0, S(0)\rangle_{2N}\,, \\ |S(t)\rangle &= \sum_{J=0}^{N} A_J e^{-iE_J t}|J, 0, S(0)\rangle_{2N}\,. \end{aligned} \tag{4.21}$$

The factors $A_J$ equal the Clebsch-Gordan coefficients:

$$A_J = C\left(\frac{N}{2}, \frac{N}{2}, \frac{N}{2}, -\frac{N}{2}, J, 0\right). \tag{4.22}$$

This can be understood considering that the initial state, $|S(0)\rangle$, equals the sum of two angular momenta with $j_1 = N/2$, $j_2 = N/2$ and projections $m_1 = N/2$, $m_2 = -N/2$. The first momentum is given by the sum of all the "up" spins, while the second is the sum of all the "down" spins.

2. *Decomposition into states of* 1 *and* $2N-1$ *spins.* It is useful to decompose the state $|S^-\rangle$ as:

$$|S^-\rangle = \sum_{J=0}^{N-1} K_{J+1/2}\left|J+\frac{1}{2}, -\frac{1}{2}, S^-\right\rangle_{2N-1}. \tag{4.23}$$

Similarly to eq. (4.22), here we have:

$$K_{J+1/2} = C\left(\frac{N-1}{2}, \frac{N-1}{2}, \frac{N}{2}, -\frac{N}{2}, J+\frac{1}{2}, -\frac{1}{2}\right) = \sqrt{\eta(N,J)}\,, \tag{4.24}$$

where $\eta(N,J)$ was given in eq. (4.5). The argument to explain (4.24) is analogous to that used to justify eq. (4.22): since the state $|S^-\rangle$ has $N-1$ spins "up" and N spins "down", it is given by the sum of two angular momenta with $j_1 = (N-1)/2$, $j_2 = N/2$ and projections $m_1 = (N-1)/2$, $m_2 = -N/2$. These momenta are the result of summing all the spins "up" and all the spins "down" separately.

Each of the states $|J, 0, S(0)\rangle_{2N}$ can be decomposed as:

$$|J,0,S(0)\rangle_{2N} = a_J\left|+\frac{1}{2}\right\rangle_1\left|J+\frac{1}{2},-\frac{1}{2},s_J^-\right\rangle_{2N-1}+b_J\left|+\frac{1}{2}\right\rangle_1\left|J-\frac{1}{2},-\frac{1}{2},s_J^-\right\rangle_{2N-1}+ \\ +c_J\left|-\frac{1}{2}\right\rangle_1\left|J+\frac{1}{2},+\frac{1}{2},s_J^+\right\rangle_{2N-1}+d_J\left|-\frac{1}{2}\right\rangle_1\left|J-\frac{1}{2},+\frac{1}{2},s_J^+\right\rangle_{2N-1}, \tag{4.25}$$

where the states $|J+1/2,+1/2,s_J^+\rangle_{2N-1}$ and $|J-1/2,+1/2,s_J^+\rangle_{2N-1}$ in the second line are related to the corresponding ones in the first line by the $z$-reflection operation. Notice that the state $|J,0,S(0)\rangle_{2N}$ has parity $(-1)^{N-J}$ under $z$-reflection. Requiring that the right hand side of eq. (4.25) have the same parity yields

$$c_J = (-1)^{N-J}a_J \qquad d_J = (-1)^{N-J}b_J\,. \tag{4.26}$$

We also note that

$$a_N = c_N = 0 \qquad b_0 = d_0 = 0\,, \tag{4.27}$$

$$|a_J|^2 + |b_J|^2 = |c_J|^2 + |d_J|^2 = \frac{1}{2}\,, \tag{4.28}$$

where the last relation represents the normalization condition.

3. *Properties of the decomposition (4.25).* We can calculate the state $\langle -1/2|_1|S(t)\rangle$ and demand that it vanish at $t=0$. This is equivalent to requiring a null probability to find the first spin initially in the down state. Combining this condition with eqs. (4.21), (4.25) and (4.26) we find

$$\sum_{J=0}^{N-1} A_J a_J (-1)^{N-J}\left|J+\frac{1}{2},+\frac{1}{2},s_J^+\right\rangle_{2N-1}+ \\ +\sum_{J=0}^{N-1} A_{J+1}b_{J+1}(-1)^{N-J-1}\left|J+\frac{1}{2},+\frac{1}{2},s_{J+1}^+\right\rangle_{2N-1} = 0\,, \tag{4.29}$$

from which it follows that

$$A_J a_J\left|J+\frac{1}{2},+\frac{1}{2},s_J^+\right\rangle_{2N-1} = A_{J+1}b_{J+1}\left|J+\frac{1}{2},+\frac{1}{2},s_{J+1}^+\right\rangle_{2N-1}, \tag{4.30}$$

and, by $z$-reflection:

$$A_J a_J\left|J+\frac{1}{2},-\frac{1}{2},s_J^-\right\rangle_{2N-1} = A_{J+1}b_{J+1}\left|J+\frac{1}{2},-\frac{1}{2},s_{J+1}^-\right\rangle_{2N-1}. \tag{4.31}$$

4. *Calculation of the survival probability $P_1$.* Let us calculate the probability $P_1(t)$ that at the time $t$ the first spin is found in its initial state, $|+1/2\rangle$:

$$P_1(t) = \left\langle S(t)\middle| +\frac{1}{2}\right\rangle_1 \left\langle +\frac{1}{2}\middle|_1 \middle| S(t)\right\rangle. \tag{4.32}$$

The combination of eqs. (4.21), (4.25) and (4.27), gives the expression of $\langle +1/2|_1|S(t)\rangle$:

$$\left\langle +\frac{1}{2}\right|_1\left|S(t)\right\rangle = \sum_{J=0}^{N-1} e^{-iE_J t} A_J a_J \left|J+\frac{1}{2},-\frac{1}{2},s_J^-\right\rangle_{2N-1} + \\ + \sum_{J=0}^{N-1} e^{-iE_{J+1} t} A_{J+1} b_{J+1} \left|J+\frac{1}{2},-\frac{1}{2},s_{J+1}^-\right\rangle_{2N-1}, \tag{4.33}$$

which, taking into account the result (4.31), simplifies to:

$$\left\langle +\frac{1}{2}\right|_1\left|S(t)\right\rangle = \sum_{J=0}^{N-1} \left(e^{-iE_J t} + e^{-iE_{J+1} t}\right) A_J a_J \left|J+\frac{1}{2},-\frac{1}{2},s_J^-\right\rangle_{2N-1}. \tag{4.34}$$

The final step to conclude the proof is the observation that for $t = 0$, eq. (4.34) should coincide with the expansion of the state $|S^-\rangle$, given in eq. (4.23). From comparison, using the equality (4.24), it follows that:

$$\begin{aligned} 2A_J a_J \left|J+\frac{1}{2},-\frac{1}{2},s_J^-\right\rangle_{2N-1} &= K_{J+1/2} \left|J+\frac{1}{2},-\frac{1}{2},S^-\right\rangle_{2N-1} \\ &= \sqrt{\eta(N,J)} \left|J+\frac{1}{2},-\frac{1}{2},S^-\right\rangle_{2N-1}. \end{aligned} \tag{4.35}$$

Combining eqs. (4.32), (4.34), (4.35) and (4.13), we find

$$P_1(t) = \sum_{J=0}^{N-1} \frac{\eta(N,J)^2}{4} |e^{-iE_J t} + e^{-iE_{J+1} t}|^2, \tag{4.36}$$

from which eq. (4.4) straightforwardly follows.

#### 4.3.3 The physical (large-$N$) limit: resummed function

To derive eq. (4.6) we first elaborate $\eta(N,J)$, eq. (4.5), in the limit $N \gg 1$. We approximate the factorials using the Stirling formula: $z! \simeq \sqrt{2\pi} e^{(z-1/2)\ln z - z}$. As a result, an exponential factor appears in the expression of $\eta(N,J)$. We expand this exponential in powers of $J/N$ and keep only the leading terms. This is justified because the various factors in eq. (4.5) are significantly different from zero only for $J \ll N$. From the procedure described, we obtain:

$$\eta(N,J) \simeq 2N^{-1}(1+J)\exp\left[-\frac{(J+1)^2}{N}\right]. \tag{4.37}$$

This function has a peak at $J = \sqrt{N/2} - 1$ and has a characteristic width $2\sigma \sim \sqrt{N}$. This fact provides an *a posteriori* justification for taking $J \ll N$ for large $N$.

As a second step, we notice that, for $N \gg 1$ the Fourier sum in eq. (4.4) is well approximated by the integral:

$$\begin{aligned} \sum_{J=0}^{N-1} \eta(N,J)\cos[(2J+2)gt] &\simeq \int_0^\infty dJ\, \frac{2}{N} J e^{-J^2/N} \cos(2Jgt) \\ &= 1 - \sqrt{\pi N} gt e^{-Ng^2t^2} \mathrm{erfi}(\sqrt{\mathrm{N}}\mathrm{gt}), \end{aligned} \tag{4.38}$$

where the expression (4.37) for $\eta$ has been used. The function erfi$(z)$ is given in eq. (4.7). From eq. (4.38) and (4.4) the final result (4.6) follows immediately.

## 5. Discussion

1. Let us compare the main features of our solution, eqs. (4.4) and (4.6), with the corresponding results for a gas of small neutrino wavepackets. In the latter case the survival probability exhibits a different dependence on time, characterized by a simple exponential form with the argument linear in $t$. To see this, consider a thought experiment in which a beam of $2N$ electron neutrinos enters the medium containing an equal mixture of electron and muon neutrinos. It is straightforward to see that, for incoherent forward scattering, the flavor composition of the beam will change according to

$$\frac{dN_e}{dt} = -\lambda(N_e - N_\mu)\,, \tag{5.1}$$
$$\frac{dN_\mu}{dt} = -\lambda(N_\mu - N_e)\,, \tag{5.2}$$

   where $\lambda$ denotes the frequency of the flavor changing collisions for a given neutrino. The solution is

$$N_e(t) = N(1+\exp(-\lambda t))\,, \tag{5.3}$$

   a simple exponential. The origin of the difference between eq. (4.6) and (5.3) lies in the different physics of the interaction. In the case of the wavepackets, the time elapsed from $t = 0$ tells us how many background neutrinos a given neutrino interacted with; the interaction time between neutrino pairs is very short. On the other hand, in the case of the interacting plane waves, any two waves in the box continuously interact with each other starting from $t = 0$. The scattering amplitude (and the effective interaction cross section) grows with time. If the conversion process were coherent, the survival probability would not depend on the particular model adopted (see section 3.2).

2. It is interesting to note that the Fourier series in eq. (4.4) describes a periodic function with period $T = \pi/g$. This periodicity is lost when a transition to the integral is made. Indeed, in eq. (4.38) the lowest frequency over which the integration is performed is zero ($J = 0$), corresponding to infinite period. It is clear that the period $T$ is much larger than the equilibration time $t_{eq}$ (see eq. (4.8)): $T = \pi\sqrt{N}t_{eq}$. Moreover, one can see that typically $T$ exceeds the age of the universe: for a volume $V = 1$ cm$^3$ we find $T \sim 10^{22}$ s. Therefore the integral form (4.6) is an accurate approximation of the exact result over time scales (at least) of the order of $t_{eq}$, and applies to all the realistic physical situations.

3. Let us comment on the range of validity of our results and their possible extensions.



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($a$) Due to the SU(2) invariance of the problem, the results are the same if we replace the flavor eigenstates $\nu_e, \nu_\mu$ with any pair of states in the flavor space, $\nu_x, \nu_y$, related to the flavor basis by an SU(2) transformation. Explicitly, if we take neutrinos initially in the states $\nu_x$ and $\nu_y$, the probability of conversion $\nu_x \to \nu_y$ does not exhibit coherent effects in the large-$N$ limit. This tells us that no coherent conversion is realized if coherence is initially absent in the system, i.e. if, for a given neutrino $\nu_i$ being in the flavor state $\nu_x$, the other neutrinos $\nu_j$ in the ensemble are *not* coherent superpositions of $\nu_x$ and $\nu_y$: $\langle \nu_i | \nu_j \rangle = 0$ or $|\langle \nu_i | \nu_j \rangle| = 1$.

($b$) Several generalizations of our calculation are possible. For instance, one could investigate how the results depend on the initial configuration of the system. In [21] we have studied the case in which coherence is initially present in the system ($|\langle \nu_i | \nu_j \rangle| < 1$) and/or the initial state is entangled. The findings were in agreement with the one-particle description, eq. (3.3).

($c$) The assumption of constant spin-spin coupling, $f_{ij} = 1$, could be relaxed to describe the case of, e.g., randomly distributed couplings or interactions which depend on the spin-spin distance or on the specific geometry of the system. These cases may be difficult or impossible to describe analytically; however we expect them to be characterized by the same incoherent character of the conversion.

($d$) The generalization to include vacuum oscillations and interaction with ordinary matter can done by adding the appropriate (well known) terms to the hamiltonian.

## 6. Conclusions

We have found that a system of many ($N \gg 1$) (forward-)interacting neutrino plane waves in a box reaches flavor equilibration after a time $t \sim t_{eq} = (\sqrt{2} G_F \sqrt{N}/V)^{-1}$ if the neutrinos are initially in flavor states. This result agrees with what is expected from incoherent scattering for this system, and therefore indicates the absence of coherent conversion effects.

Although our results were obtained under specific simplifying conditions (plane-waves in a box, constant couplings, forward scattering only, etc.), they can be used to draw important general conclusions on neutrino conversion in a neutrino background. The absence of coherent conversion has general character, since it does not depend on the size of the neutrino wavepackets (see section 3.1). It contrasts with ref. [22] and agrees with the prediction of the traditional — and widely applied — single-particle description. This tells us that there is no reason to believe in a breakdown of this description and that, therefore, its applications have to be considered valid. The significance of this statement is clear in consideration of the important physical problems, in cosmology and in stellar physics, in which the neutrino-neutrino coherent scattering plays a role.

Finally, we remark that, in principle, the analytical solution we have found for a system of interacting spins may be interesting on its own and may find applications beyond the neutrino evolution problem considered here.



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